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Notes on the efficiency of propulsion of bodies in waves

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Bodies that absorb, reflect or generate wave energy are submitted to mean forces. For moving bodies the mean forces in the direction of motion contribute to the drag or propulsion of the body. For flexible and deformable slender bodies swimming in waves at a constant forward velocity U normal to the crests of the waves, the mean rate of working $\overline{W}$ and the mean thrust $\overline{T}$ are evaluated. When the waves are assumed to be not significantly affected by the swimming slender bodies it is found that the Froude efficiency of propulsion for cases without shedding of vorticity is invariably given by $U/(U+c)$, $U+c$ being the phase velocity of the waves with respect to the body. The result remains valid when shedding small amounts of vorticity. $\overline{T}$ is obtained as the result of the radiation stress, and is proportional to $\overline{W}$.

The same efficiency can be realized by two-dimensional bodies oscillating in regular trains of two-dimensional waves. It is also valid for wave-making boats. For three-dimensional cases $U/(U+c)$ represents the upper limit when the outgoing waves are properly beamed. Actuator surfaces with constant loading will be interpreted as vortex wavemakers.

1. Introduction

An unbounded inviscid incompressible fluid offers no resistance to steady translational motion of a rigid body. This is known as d'Alembert's paradox. For a body that, in addition to the steady translation, deforms periodically one has the same paradox for the mean value of the resistance over one period (and also over a period of time tending to infinity). The kinetic energy in the flow varies periodically, and the mean rate of working by the body against the pressure in the fluid is equal to zero. In a real fluid with viscosity a self-propelling body at constant (or periodic) forward speed must experience mean values of the thrust and drag that cancel. Undulatory modes of animal locomotion in water for the high-Reynolds-number range (say from $Re = 10^4$ onwards) involve the generation of a mean reaction force, which can be estimated with an inviscid flow model that is valid outside the boundary layer. At sharp trailing edges an appropriate Kutta condition may be imposed. At sharp trailing edges of operative tails and fins vorticity is being shed, and the mean rate of shedding energy into the vortex wake represents the wasted part of the mean rate of working by the body. In such a theory for a swimming body in a uniform stream a mean thrust that balances the mean drag is necessarily associated with a vortex wake.

When the oncoming flow is non-uniform, as in the case of a river streaming over an irregular bed, or in cases with free surface waves, a swimming body may experience thrust or drag without vortex shedding. Bodies that absorb, reflect or generate wave energy are submitted to mean forces. Hence in generating propulsion in non-uniform flow conditions one may distinguish two mechanisms, one that involves vortex shedding and another one that does not depend on vortex shedding. The mean thrust $\overline{T}$ on the body, doing useful work at a rate $\overline{T}U$ (where U is the swimming speed), and the mean rate of working $\overline{W}$ by the body can be decomposed into two far-field components, a vortex and a wave component. The present paper is mainly concerned with the Froude efficiency $\eta = \overline{T}U/\overline{W}$ in several examples without vortex shedding.

In §2 the case of slender bodies in a regular train of surface waves is discussed. In an earlier version of slender-body theory worked out by Coene (1975) for swimming in waves, the cross-sections of the body were assumed to be undeformable during the swimming motions. This restriction is removed in the present paper so that peristaltic deformations (either passive or active) may be included. On the assumption that the situation at the tail dominates the generation of thrust, it was shown in the earlier paper that, from the volume that is effectively being swept by the tail, one half of the kinetic energy present due to the waves can be extracted and made available for propulsion. Without suggesting that animals should neglect their tails, we shall now concentrate on the body–wave interactions, which are complementary to the tail terms and also exist when the body has no operative tail involving the shedding of vorticity. The mean thrust or drag on the body then results from the radiation stress in the ambient wave, and is found to be proportional to the mean rate of working. The slender-body results are obtained on the assumption that the oncoming wave is not significantly affected as far as the cross-flow induced at the mean swimming depth is concerned. The towing-tank experiments described by Coene (1977) tend to confirm the validity of the present variety of slender-body theory for a not-so-deeply submerged rigid slender delta-wing-like body. In these experiments the body was provided with strain-gauge dynamometers at two pivots, and was towed through a regular head sea while the body was forced to carry out heaving and pitching oscillations. Upon proper adaptation of these oscillations to the oncoming waves, thrust could be generated at efficiencies predicted from slender-body theory.

In the two-dimensional cases, discussed in §3, the body may swim in its own waves also. In the absence of forward speed, the two-dimensional results correspond with those for wave-power machines and stationary vessels as discussed by Longuet-Higgins (1977). The results of §3 will be applied to estimate the Froude efficiency of his wave-making boat. In the two-dimensional cases, the calculations are based on far-field bilinear potential-flow results for bodies carrying a constant circulation. The two-dimensional results are complementary to those of Sparenberg (1976) and Wu (1972), where the shedding of vorticity plays an essential role in the generation of propulsion and the extraction of wave energy.

Several examples will be worked out in which the Froude efficiency of propulsion is invariably given by $U/(U+c)$. In the slender-body case the result is obtained by direct calculation of the mean thrust and the mean rate of working. In the two-dimensional case the far-field point of view permits the evaluation of the efficiency without bothering about the details at the body. In more general three-dimensional cases there will be losses to the sides. The beaming will be less than perfect, and the Froude efficiency of wavemaking and wave interaction cannot exceed $U/(U+c)$.

2. The slender-body case

In an earlier paper (Coene 1975) the author discussed the swimming of slender bodies in regular trains of waves. The propulsion problem involving the interaction

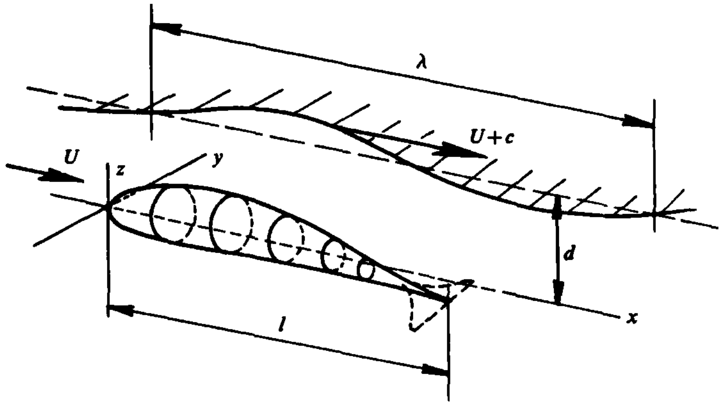

FIGURE 1. A slender actuator body in waves. c is positive in a head sea and negative in a following sea.

of the swimming motions with the oncoming waves was worked out on the assumption that the situation at the tail dominates the mean rate of working, the mean thrust and thereby the efficiency of propulsion. It was shown that, upon proper adaptation of the voluntary part of the tail motions to the lateral component of the orbital velocity in the waves, energy can be extracted from the waves. Propulsion at a higher Froude efficiency than in the absence of waves can thus be generated in head seas as well as following seas. Whenever the tail is operative, vorticity is being shed into the wake. When the tail is inoperative or when the body has no tail with a sharp trailing edge, the body may still interact with the waves and generate a mean thrust or drag, while the mean rate of working by the body against the pressure forces in the water does not vanish. In the present paper we shall focus our attention on these body–wave interactions, which are complementary to the tail terms discussed earlier. Moreover, the earlier restriction as to the undeformability of the sections of the flexible slender body will be removed, thus allowing for peristaltic deformations. The results for the hydrodynamic forces on the body will be manipulated in such a way that a physical interpretation of the body terms as line-distributions of Lagally forces is possible. This, in turn, leads to a simple result for the Froude efficiency of propulsion in waves. The bodies we have in mind have the same symmetry as cetacean mammals, and the swimming motions are parallel to the plane of symmetry.

A Cartesian coordinate system (x, y, z) performs a steady translation with the mean position of the swimming body at a mean depth d with a velocity U in the $-x$-direction with respect to the water at rest far from the surface. A regular train of two-dimensional waves of amplitude a, wavelength λ and phase velocity $U+c$ with respect to the (x, y, z)-system in the $+x$-direction is given by the velocity potential

$$\phi_w(x, z, t) = ac \exp[2\pi\lambda^{-1}(z-d)] \cos\{2\pi\lambda^{-1}[x-(U+c)t]\}. \tag{2.1}$$

The velocity c is positive in a head sea and negative in a following sea. $U+c = 0$ corresponds to the steady wave-riding case. The body is assumed to be slender. With ϵ as a small slenderness parameter, the lateral dimensions are $O(\epsilon l)$ and they are slowly varying functions of x and t. The body is considered to be 'stretched straight' when, in uniform flow conditions and without free-surface effects, the deformations of the cross-sections are such that no resultant normal force except buoyancy acts on any cross-section. With respect to these stretched straight positions for bodies performing peristaltic deformations, the body also carries out smoothly varying swimming motions $h(x,t)$, $O(\epsilon l)$, in the z-direction. Thus, defining the stretched straight positions by $F(x,y,z,t)=0$ gives a swimming body $F(X,Y,Z,T)=0$ upon introduction of the (X,Y,Z,T)-system with

$$X=x, \quad Y=y, \quad Z=z-h(x,t), \quad T=t. \tag{2.2}$$

With ϕ^*_{wz} representing the z-component of the orbital velocity due to the waves at the mean swimming depth, the resultant cross-flow $w^*(x,t)$ can be expressed as

$$w^*(x,t)=\frac{\partial h}{\partial t}+U\frac{\partial h}{\partial x}-\phi^*_{wz}=w-\phi^*_{wz}. \tag{2.3}$$

All velocity components in (2.3) are assumed to be $O(\epsilon U)$ and smoothly varying. The velocity potential $\Phi(X,Y,Z,T)$ satisfies the transformed Laplace equation $(\partial/\partial X-(\partial h/\partial X)\,\partial/\partial Z)^2\,\Phi+\partial^2\Phi/\partial Y^2+\partial^2\Phi/\partial Z^2=0$. By virtue of the slender-body assumptions, Φ may be decomposed as follows:

$$\Phi(X,Y,Z,T)=UX+\Phi_w(X,Z,T)+\Phi_0(X,Y,Z,T)+\Psi(X,Y,Z,T). \tag{2.4}$$

In (2.4), Φ_w follows from (2.1) with (2.2). Φ_0 is the perturbation potential of the body with $w^*=0$, 'stretched straight' along the X-axis. In Φ_0 the presence of the free surface is neglected. Starting from a known stretched straight slender-body solution $\phi_0(x,y,z,t)$, which satisfies $\partial^2\phi_0/\partial y^2+\partial^2\phi_0/\partial z^2=0$ near the body, one obtains $\Phi_0(X,Y,Z,T)$, which satisfies $\partial^2\Phi_0/\partial Y^2+\partial^2\Phi_0/\partial Z^2=0$ near the body, by replacing z in ϕ_0 by Z, i.e. by taking the same function $\Phi_0=\phi_0$. Ψ is the potential proportional to w^*, with the strip assumption

$$\Psi(X,Y,Z,T)=w^*(X,T)\,\phi(Y,Z;X,T), \tag{2.5}$$

where ϕ satisfies $\partial^2\phi/\partial Y^2+\partial^2\phi/\partial Z^2=0$ and $\phi\to 0$ at infinity as in an unbounded mass of water. The errors that may arise near the ends of the body are assumed to be local.

The free surface effects of the type that would also arise in the absence of the ambient wave Φ_w are not accounted for in (2.4). For a given body in a given swimming mode, the errors involved are exponentially dependent on the mean swimming depth d. Thus neglecting the free-surface effects in Φ_0 and Ψ amounts to assuming that d not be too small. Neglecting these free-surface effects brings out the interaction of the swimming body with the oncoming waves, and turns out to be equivalent to assuming that the ambient wave Φ_w not be significantly affected at the free surface. This amounts to ignoring the free-surface origin of the ambient non-uniformity in which the body is swimming. Not surprisingly then, the results obtained in this section are independent of the dispersion relation satisfied by the gravity waves (2.1), and they also apply to swimming in non-uniformities associated with solid boundaries such as a wavy river bed.

The boundary conditions for Φ_0 and Ψ are obtained by setting equal to zero the rate of change $\mathrm{D}F/\mathrm{D}T$ following a particle of water. Omitting products of small quantities, one obtains as the boundary conditions for Φ_0 and Ψ at the body:

$$\frac{\partial F}{\partial T}+U\frac{\partial F}{\partial X}+\frac{\partial \Phi_0}{\partial Y}\frac{\partial F}{\partial Y}+\frac{\partial \Phi_0}{\partial Z}\frac{\partial F}{\partial Z}=0 \quad \text{at } F=0, \tag{2.6}$$

$$\frac{\partial \Psi}{\partial Y}\frac{\partial F}{\partial Y}+\left(\frac{\partial \Psi}{\partial Z}-w^*\right)\frac{\partial F}{\partial Z}=0 \quad \text{at } F=0. \tag{2.7}$$

From Bernoulli's equation, one obtains in the (X,Y,Z,T)-system, with a decomposition for the pressure

$$p=p_0+p_1+p_2+p_3, \tag{2.8}$$

up to and including orders ϵ^2 and $\epsilon^2 \log \epsilon$ and omitting the hydrostatic part,

$$\left.\begin{aligned} p_0 &= \text{const} - \rho\left[\left(\frac{\partial}{\partial T} + U\frac{\partial}{\partial X}\right)\Phi_0 + \frac{1}{2}\left\{\left(\frac{\partial \Phi_0}{\partial Y}\right)^2 + \left(\frac{\partial \Phi_0}{\partial Z}\right)^2\right\}\right], \\ p_1 &= -\rho\left\{\left(\frac{\partial}{\partial T} + U\frac{\partial}{\partial X}\right)\Psi + \frac{\partial \Phi_0}{\partial Y}\frac{\partial \Psi}{\partial Y} + \frac{\partial \Phi_0}{\partial Z}\left(\frac{\partial \Psi}{\partial Z} + \frac{\partial \Phi_w}{\partial Z} - w\right)\right\}, \\ p_2 &= -\rho\left\{\left(\frac{\partial}{\partial T} + U\frac{\partial}{\partial X}\right)\Phi_w - w\frac{\partial \Phi_w}{\partial Z} + \frac{1}{2}\left(\frac{\partial \Phi_w}{\partial X}\right)^2 + \frac{1}{2}\left(\frac{\partial \Phi_w}{\partial Z}\right)^2\right\}, \\ p_3 &= \rho\left\{\left(w - \frac{\partial \Phi_w}{\partial Z}\right)\frac{\partial \Psi}{\partial Z} - \frac{1}{2}\left(\frac{\partial \Psi}{\partial Y}\right)^2 - \frac{1}{2}\left(\frac{\partial \Psi}{\partial Z}\right)^2\right\}. \end{aligned}\right\} \tag{2.9}$$

The lateral force

We first discuss the lateral force $L(X, T)$ per unit length in the Z-direction. By virtue of the definition of Φ_0, p_0 does not contribute to L. The first term of p_1 yields a contribution, $O(\epsilon^3)$,

$$L_1 = -\rho\left(\frac{\partial}{\partial T} + U\frac{\partial}{\partial X}\right)\{w^* A(X, T)\}, \tag{2.10}$$

where $A(X, T)$ is the virtual mass per unit length of a cylinder moving in the Z-direction, and is defined by the contour integral

$$A(X, T) = \oint_{X,T} \phi(Y, Z; X, T)\,\mathrm{d}Y, \tag{2.11}$$

with ϕ defined in (2.5).

Upon expansion of Φ_w with respect to Z, one finds from the first term in p_2, to $O(\epsilon^3)$,

$$L_2 = \rho S(X, T)\left(\frac{\partial}{\partial T} + U\frac{\partial}{\partial X}\right)\Phi^*_{wZ}, \tag{2.12}$$

where $S(X, T)$ is the area of the cross-section. Combining (2.10) and (2.12) and rewriting the result in the variables x, t and ϕ^*_w, we thus obtain, to $O(\epsilon^3)$,

$$L(x, t) = -\rho\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\{w^* A(x, t)\} + \rho S(x, t)\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi^*_{wz}. \tag{2.13}$$

Upon introducing the coordinate system (x_1, y_1, z_1, t_1) fixed in the water at rest far from the surface, with

$$x_1 = x - Ut, \quad y_1 = y, \quad z_1 = z, \quad t_1 = t, \tag{2.14}$$

one has

$$\frac{\partial}{\partial t_1} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}, \tag{2.15}$$

which is also a first approximation to the material time derivative, following particles of water. Thus the first term in (2.13) can be interpreted as the reaction force on the body per unit length due to the time rate of change of lateral momentum of water slices of unit thickness. Rewriting the last term in (2.13) in the form

$$\rho S(x_1 + Ut_1, t_1)\frac{\partial}{\partial z_1}\left(\frac{\partial \phi^*_w}{\partial t_1}\right) \tag{2.16}$$

shows that this term is due to the vertical component of the pressure gradient in the waves. Even with undeformable cross-sections, A and S are time-dependent with respect to t_1. Thus the result for the lateral force, obtained earlier for undeformable cross-sections, is found to remain valid in the more general case of deformable cross-sections.

The lateral force can also be interpreted as a Lagally force upon rewriting (2.13), using (2.3) and (2.14), in the form

$$L(x_1, t_1) - \rho \frac{\partial}{\partial t_1}(wS) = -\rho \frac{\partial}{\partial t_1}\{w^*(A+S)\} - \rho \phi^*_{wz} \frac{\partial}{\partial t_1} S. \tag{2.17}$$

The second term on the left-hand side of (2.17) represents the inertial effect of water slices (of unit thickness) displaced by the body in the form of the rate of change of their lateral momentum. The first term on the right-hand side is minus the rate of change of lateral momentum of the cross-flow induced by doublets with a vertical axis and moment proportional to $w^*(A+S)$. The last term is the vertical component of the force on a source of strength $\partial S/\partial t_1$ in a vertical velocity field ϕ^*_{wz}.

The mean thrust

The mean thrust may be expressed as

$$\overline{T} = \iint_S \overline{p\,\mathrm{d}y\,\mathrm{d}z} = \iint_S \overline{p\,\mathrm{d}Y\,\mathrm{d}Z} + \int_0^l \overline{L\frac{\partial h}{\partial X}}\mathrm{d}X. \tag{2.18}$$

In Appendix A the $O(\epsilon^3)$ and $O(\epsilon^4)$ contributions to the mean thrust are derived. The result can be decomposed in a tail term $\overline{T}_t$ and a body term $\overline{T}_b$:

$$\overline{T} = \overline{T}_t + \overline{T}_b, \tag{2.19}$$

with

$$\overline{T}_t = \tfrac{1}{2}\rho\left[\overline{A\left\{\left(\frac{\partial h}{\partial t} - \phi^*_{wz}\right)^2 - U^2\left(\frac{\partial h}{\partial x}\right)^2\right\}}\right]_{x=l} \tag{2.20}$$

and

$$\overline{T}_b = \rho \int_0^l \overline{(\phi^*_{wx} + h\phi^*_{wzx})\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)S}\,\mathrm{d}x + \rho\int_0^l \overline{(A+S)\,w^*\phi^*_{wzx}}\,\mathrm{d}x. \tag{2.21}$$

The tail term (2.20) was discussed in some detail by Coene (1975) on the assumption that the situation at the tail dominates the generation of propulsion. We shall now concentrate on the body term (2.21), which is of interaction type and vanishes in the absence of the ambient wave. It is complementary to the tail term, but also arises for bodies without an operative tail that involves vortex shedding. A non-vanishing interaction (2.21) indicates that by virtue of the presence of the ambient wave thrust or drag can be generated without entailing vorticity. The first integral on the right-hand side of (2.21) was not retained in the 1975 paper. The steady wave-riding case and peristaltic deformations were not included there. On the other hand, as shown in Appendix A, the term proportional to $h\phi^*_{wzx}$ yields an $O(\epsilon^4)$ contribution in the thrust for undeformable sections (with $\partial S/\partial t = 0$) also, and it would have been consistent to retain it on the assumption that h is $O(\epsilon l)$. The integrand in the first integral of (2.21) can be interpreted as the mean force, per unit length, on a source of strength $(\partial/\partial t + U\partial/\partial x)S)$ in a velocity field with a horizontal component $\phi^*_{wx} + h\phi^*_{wzx}$. The second integral in (2.21) is the mean horizontal force on a distribution of doublets with vertical axis and a moment $w^*(A+S)$ in a velocity field with vertical gradient ϕ^*_{wzx}.

The $O(\epsilon^3)$ contribution vanishes in the mean for undeformable sections and oscillatory ϕ^*_{wx}. In the case of steady wave-riding, ϕ^*_{wx} is time-independent, and a favourable position in the waves leads to a positive $O(\epsilon^3)$ thrust

$$T_b = \rho \int_0^l U \frac{\partial S}{\partial x} \phi^*_{wx} \, dx + O(\epsilon^4). \tag{2.22}$$

The possibility of favourable positioning is not restricted to the non-uniformities associated with regular waves of the type (2.1). The case of bow-wave-riding porpoises is well known. On the other hand a term like (2.22) depends only on the lengthwise pressure distribution in the non-uniformity. Obviously, such $O(\epsilon^3)$ contributions may also be exploited by fishes for a relative state of rest in a river with non-uniformities fixed with respect to the bed of the river. The first term in (2.21) also shows that, in the cases of unsteady wave-riding, peristaltic deformations correlated with the pressure wave moving along the body may lead to a mean thrust or drag $O(\epsilon^3)$ whenever the lateral velocities due to the peristaltic deformations are $O(\epsilon U)$. In the expressions for the thrust both A and S are $O(\epsilon^2 l^2)$, but the results remain valid for planar bodies with $S \ll A$, or even in the waving-plate case with $S \equiv 0$ (with, in addition, $\partial A/\partial x \geqslant 0$ in order to avoid free vorticity for $x < l$), without the need to introduce higher-order terms for the contributions proportional to S.

The mean rate of working

The mean rate of working by the body against the pressure forces can be expressed as

$$\overline{W} = \int_0^l \overline{p \frac{\partial S}{\partial t}} \, dx - \int_0^l \overline{L \frac{\partial h}{\partial t}} \, dx. \tag{2.23}$$

The first term is due to the peristaltic deformations, the second one is readily obtained upon substitution of the lateral force (2.13). Peristaltic deformations do no work in the mean in the case of uniform oncoming flow, but they can do work against the pressure due to the ambient wave. Expressing the term p_2 at $z = h(x,t)$ (also appearing in (2.9) in the (X, Y, Z, T)-system with Φ_w) in the (x, y, z, t)-system with ϕ^*_w yields

$$p_2 = -\rho \left[\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \phi^*_w + h \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \phi^*_{wz} + \tfrac{1}{2} (\phi^{*2}_{wx} + \phi^{*2}_{wz}) \right]. \tag{2.24}$$

The last term is independent of x and t, and cannot contribute to (2.23). Substituting (2.13) and (2.24) in (2.23) and assuming $S(0) = 0$, $S(l) = 0$, $A(0) = 0$ but $A(l) \neq 0$ at the trailing edge gives

$$\overline{W} = \overline{W}_t + \overline{W}_b, \tag{2.25}$$

with

$$\overline{W}_t = \rho U \left[\overline{A \frac{\partial h}{\partial t} w^*} \right]_{x=l} \tag{2.26}$$

for the tail term and

$$\overline{W}_b = -\rho \int_0^l \overline{\left[\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \phi^*_w + h \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \phi^*_{wz} \frac{\partial S}{\partial t} \right]} \, dx$$
$$+ \rho \int_0^l \overline{(A+S) \frac{\partial w}{\partial t} \phi^*_{wz}} \, dx + \rho \int_0^l \overline{\left(\frac{\partial S}{\partial t} + U \frac{\partial S}{\partial x} \right) \frac{\partial h}{\partial t} \phi^*_{wz}} \, dx \tag{2.27}$$

for the body term.

The efficiency of propulsion

We begin by considering the $O(\epsilon^3)$ contributions in $\overline{T}$ and $\overline{W}$:

$$\left.\begin{aligned}\overline{T}_{\mathrm{b}} &= \rho \int_0^l \overline{\phi^*_{wx}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)S}\,\mathrm{d}x+O(\epsilon^4),\\ \overline{W}_{\mathrm{b}} &= -\rho\int_0^l \overline{\frac{\partial S}{\partial t}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\phi^*_w}\,\mathrm{d}x+O(\epsilon^4).\end{aligned}\right\} \tag{2.28}$$

Non-vanishing time-averaged $O(\epsilon^3)$ terms may occur when the peristaltic deformations are correlated with the leading-order term in the pressure due to the waves. The wave potential (2.1) depends on the variable $x-(U+c)\,t$, so one has, invariably,

$$\phi_{wt} = -(U+c)\,\phi_{wx}. \tag{2.29}$$

For periodic S and with $S(0) = 0$, $S(l) = 0$, $\overline{W}_{\mathrm{b}}$ in (2.28) may be rewritten as

$$\overline{W}_{\mathrm{b}} = -\rho\int_0^l \overline{\phi^*_{wt}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)S}\,\mathrm{d}x+O(\epsilon^4),$$

which, by virtue of (2.28) and (2.29), may be expressed as

$$\overline{W}_{\mathrm{b}} = (U+c)\,\overline{T}_{\mathrm{b}}+O(\epsilon^4). \tag{2.30}$$

Obviously (2.30) remains valid in the case of steady wave-riding, with $U+c=0$, where an $O(\epsilon^3)$ thrust (or drag) results from the steady pressure gradient in the wave without any work being done by the body. It may be observed that these $O(\epsilon^3)$ terms are independent of the swimming motions $h(x,t)$. On the other hand, the $O(\epsilon^4)$ contributions to $\overline{T}$ and $\overline{W}$ do depend on the motions $h(x,t)$.

As shown above, the $O(\epsilon^3)$ terms satisfy the simple relation (2.30), and, it will be shown, the $O(\epsilon^4)$ terms turn out to satisfy the same relation for oscillatory swimming motions and peristaltic deformations at the frequency of encounter $\omega_{\mathrm{e}} = (2\pi/\lambda)\,|U+c|$. From (2.29) one readily derives, for oscillatory swimming motions characterized by the frequency of encounter,

$$\overline{\frac{\partial w}{\partial t}\phi^*_{wz}} = \overline{\frac{\partial}{\partial t}(w\phi^*_{wz})}-\overline{w\phi^*_{wzt}} = (U+c)\,\overline{w\phi^*_{wzx}}, \tag{2.31}$$

$$\overline{\frac{\partial h}{\partial t}\phi^*_{wz}} = \overline{\frac{\partial}{\partial t}(h\phi^*_{wz})}-\overline{h\phi^*_{wzt}} = (U+c)\,\overline{h\phi^*_{wzx}}. \tag{2.32}$$

Moreover, with $w^* = w-\phi^*_{wz}$, one has

$$\overline{w\phi^*_{wzx}} = \overline{w^*\phi^*_{wzx}}+\overline{\frac{1}{2}\frac{\partial}{\partial x}(\phi^{*2}_{wz})} = \overline{w^*\phi^*_{wzx}}. \tag{2.33}$$

On the other hand, if $S(x,t)$ is decomposed as $S_0(x)+S_1(x,t)$, where S is periodic at the frequency of encounter, then terms in S_1 do not contribute to the mean values $O(\epsilon^4)$. The same is true for the apparent mass terms proportional to A. Thus, using (2.31), (2.32) and (2.33), one obtains

$$\overline{W}_{\mathrm{b}} = (U+c)\,\overline{T}_{\mathrm{b}}+O(\epsilon^5), \tag{2.34}$$

which extends (2.30) to include the $O(\epsilon^4)$ terms. In (2.34) $\overline{T}_{\mathrm{b}}$ is obtained in the form of a radiation stress on an actuator body. In the case of unsteady interaction, with $U+c \neq 0$, there is a unique relation between the mean force on the body in the direction of propagation of the wave and the mean rate of working by the body.

For $U+c \neq 0$ it also follows from (2.34) that an elastic slender body in passive recoil will experience no mean thrust or drag force. For $U+c < 0$ a positive mean thrust is obtained at a negative mean rate of working, implying that in this velocity régime damping is favourable for propulsion. On the other hand, for $U+c > 0$, energy extraction by the body is invariably associated with drag. It should be noted that, by contrast, interactions involving the shedding of vorticity by an operative tail can be used to generate thrust by energy extraction in any velocity régime, as shown by Coene (1975).

The result (2.34) implies a simple expression for the Froude efficiency of propulsion. Including terms $O(\epsilon^3)$ and $O(\epsilon^4)$ in $\overline{T}_b$ and $\overline{W}_b$, one has, invariably,

$$\eta_b = \frac{\overline{T}_b U}{\overline{W}_b} = \frac{U}{U+c}. \tag{2.35}$$

It follows from (2.35) that in a head sea with $c > 0$ one has $\eta_b < 1$ from the body terms. In a following sea with $0 < U+c < U$ one has $\eta_b > 1$. In the following sea with $U+c < 0$ positive $\overline{T}_b$ can be obtained at negative $\overline{W}_b$ only, as observed in the discussion of (2.34). In the special case of steady wave riding with $U+c = 0$, an $O(\epsilon^3)$ thrust may be derived, by proper positioning in the wave, at 'infinite efficiency', i.e. without doing any work.

In Coene (1975) it was shown that the interactions of the motions of the tail with the oncoming waves can lead to $\eta_t > 1$ in all velocity régimes. Comparison of the 'tail terms' and the 'body terms' indicates that in principle the body terms tend to be less attractive than the tail terms in a head sea. In a following sea, however, the body terms may be significant, especially in those cases involving $O(\epsilon^3)$ contributions to the propulsion.

A unified treatment of the various contributions is beyond the scope of the present paper. Before proceeding to the two-dimensional case, however, we shall discuss a special swimming problem for a slender body with a weakly operative tail in some detail. The equations of motion of the body are

$$\left.\begin{aligned} \rho\int_0^l S\frac{\partial^2 h}{\partial t^2}\,\mathrm{d}x &= \int_0^l L\,\mathrm{d}x, \\ \rho\int_0^l xS\frac{\partial^2 h}{\partial t^2}\,\mathrm{d}x &= \int_0^l xL\,\mathrm{d}x, \end{aligned}\right\} \tag{2.36}$$

with L given by (2.13).

A particular solution $\tilde{h}(x,t)$ of (2.36) is obtained by equating the local time rate of change of the lateral momentum of the body and the local lateral force exerted on the body by the time-varying part of the pressure of the water:

$$\rho S\frac{\partial^2 \tilde{h}}{\partial t^2} = \tilde{L}. \tag{2.37}$$

Solutions $\tilde{h}$ of (2.37) represent a passive flexible recoil mode involving no bending moments in the body. Restricting ourselves to the cases with undeformable sections, (2.37) implies that the local mean rate of working is equal to zero:

$$-\overline{\frac{\partial \tilde{h}}{\partial t}\tilde{L}(x,t)} = -\rho S\frac{1}{2}\,\overline{\frac{\partial}{\partial t}\left(\frac{\partial \tilde{h}}{\partial t}\right)^2} = 0. \tag{2.38}$$

Upon integration with respect to x, one has

$$\overline{\tilde{W}}_b = 0, \tag{2.39}$$

and by virtue of (2.34) one has, for $U+c \neq 0$,

$$\bar{T}_{\mathrm{b}} = 0, \tag{2.40}$$

for the flexible recoil mode defined by (2.37).

We now note that (2.36) can be satisfied, rather trivially, by a waving plate, with $S(x) = 0$ and vanishing cross-flow:

$$w^*(x,t) = \frac{\partial \tilde{h}}{\partial t} + U\frac{\partial \tilde{h}}{\partial x} - \phi^*_{wz} = 0. \tag{2.41}$$

A waving plate satisfying (2.41) would cause no perturbations in the flow. Thus an inoperative tail behaving like a waving plate with vanishing cross-flow may have a sharp trailing edge without shedding vorticity.

The general case of swimming motions in waves can be decomposed as

$$h(x,t) = \tilde{h}(x,t) + f(x,t), \tag{2.42}$$

where $\tilde{h}$ is the flexible recoil mode, being passive and a solution of (2.36), while f represents the voluntary part, which may or may not be correlated with the oncoming waves. The equations of motion (2.36) are linear in ϕ^*_w, so f needs to be a homogeneous solution of (2.36) in the absence of waves only.

With the additional assumption $f \ll \tilde{h}$ at the tail, and retaining terms linear in f only, the tail terms (2.21) and (2.26) yield

$$\overline{T}_{\mathrm{t}} = -\rho U\left[\overline{A\frac{\partial \tilde{h}}{\partial x}\left(\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x}\right)}\right]_{x=l}, \tag{2.43}$$

$$\overline{W}_{\mathrm{t}} = \rho U\left[\overline{A\frac{\partial \tilde{h}}{\partial t}\left(\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x}\right)}\right]_{x=l}. \tag{2.44}$$

Using (2.29) and (2.41), one finds, at $x=l$,

$$\frac{\partial \tilde{h}}{\partial t} = -(U+c)\frac{\partial \tilde{h}}{\partial x}. \tag{2.45}$$

Substituting (2.45) in (2.44) then shows, upon comparison with (2.43), that

$$\overline{W}_{\mathrm{t}} = (U+c)\overline{T}_{\mathrm{t}}. \tag{2.46}$$

Thus, rather surprisingly, even with a weakly operative tail, involving the shedding of vorticity, the efficiency of propulsion is still given by $U/(U+c)$. As shown by Coene (1975), the other possibilities with $f \approx \tilde{h}$ and $f \gg \tilde{h}$ at the tail yield other efficiencies for the tail terms.

Discussion

The results of the present section indicate that $U/(U+c)$ is important for the evaluation of the propulsion of slender bodies interacting with waves. It may be of interest to note that this number also has a simple geometrical significance. In a correlated swimming mode all parts of the body will oscillate at the frequency of encounter $\omega_{\mathrm{e}} = (2\pi/\lambda)(|U+c|)$. In the coordinate system (2.14) fixed to the water at rest far from the surface, the wavelength μ of the tracks of the body sections follow from the correlation condition $(|U+c|)/\lambda = U/\mu$, implying

$$\frac{\mu}{\lambda} = \frac{U}{|U+c|} = |\eta_{\mathrm{b}}|, \tag{2.47}$$

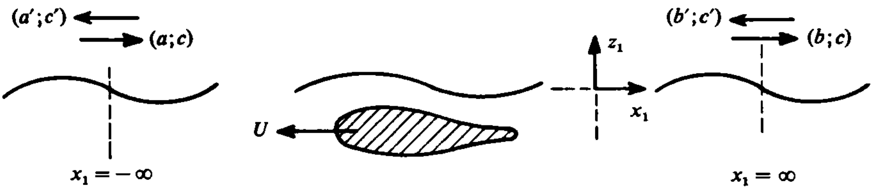

FIGURE 2. Two-dimensional swimming body with incoming and outgoing waves at infinity.

thus equating the efficiency of propulsion to the ratio of two wavelengths. Comparing the expression (2.17) for the momentaneous $O(\epsilon^3)$ lateral force and (2.21) for the mean thrust $O(\epsilon^3)$ and $O(\epsilon^4)$ shows that they involve the same line distribution of sources and doublets at $z = h(x,t)$. The $O(\epsilon^3)$ and $O(\epsilon^4)$ mean thrust $\overline{T}$ and the mean rate of working $\overline{W}$ depend on the geometry of the body sections only as far as the surface area S and the virtual mass proportional to A are concerned. The slender body may be replaced by an equivalent slender body with elliptic sections with horizontal axis b and vertical axis τ related to S and A by $S = \frac{1}{4}\pi\tau b$ and $A = \frac{1}{4}\pi b^2$. In evaluating the efficiency of propulsion, it was found that for cases without shedding vorticity (or only relatively small amounts of it) the details of the geometry and the swimming mode (of course remaining within the scope of the slender-body assumptions) are irrelevant. The Froude efficiency is invariably given by $U/(U+c)$.

It should be observed that the near-field calculations and the distinction in body terms and tail terms is possible only by virtue of the slender-body assumptions. In the present context with ambient waves the use of a far-field concept such as 'wave drag' may be confusing, but some remarks on it are in order. The steady wave drag (with the body rigidly constrained and towed steadily at speed U and depth d), which also arises in the absence of ambient waves, was not accounted for, but may be given the same status as the viscous drag, which must also be balanced in the mean by the thrust generated by periodic swimming. The thrust or drag and the rate of working associated with unsteady wavemaking of the type also arising in the absence of ambient waves will be discussed in §3. There it will be shown that the efficiency of propulsion by wavemaking has a definite value in the two-dimensional case ($U/(U+c)$!) which cannot be exceeded in the three-dimensional case.

3. The two-dimensional case

We consider two-dimensional potential flows around bodies swimming at constant mean horizontal velocity U with respect to the system (x_1, z_1, t_1) fixed to the deep water at rest far from the surface. The bodies are assumed to perform periodic swimming motions without shedding vorticity. Bodies such as hydrofoils with sharp trailing edges may be allowed to carry a non-vanishing constant bound circulation.

As indicated in figure 2, there is an incoming head sea of amplitude a and phase velocity c at $x_1 = -\infty$. At the same time there is an outgoing wave of amplitude a' and phase velocity c'. At $x_1 = \infty$ there is an outgoing wave of amplitude b and phase velocity c. Finally, at $x_1 = \infty$ we have an incoming wave of amplitude b' and phase velocity c'.

The introduction of the four waves permits a rather general discussion of the two-dimensional swimming problem in waves. The flow is periodic in a reference system moving steadily (at constant U) with the swimming body, Far from the body the incoming and outgoing waves are also periodic in the (x_1, z_1, t_1)-system.

By virtue of the periodicity of the flow it is possible to establish unique relations between the mean thrust $\overline{T}_b$ on the body, the mean rate of working $\overline{W}_b$ and the waves far from the body.

As shown in Appendix B, the mean excess momentum flux for the superposition of waves a and a' at $x_1 = -\infty$ is given by

$$\overline{F}(x_1 = -\infty) = \tfrac{1}{4}\rho g(a^2 + a'^2). \tag{3.1}$$

Similarly, one has

$$\overline{F}(x_1 = \infty) = \tfrac{1}{4}\rho g(b^2 + b'^2). \tag{3.2}$$

The mean rate at which momentum $\overline{I}$ in the x_1 direction in the flow is being increased follows from (B 14) as

$$U\overline{I} = \tfrac{1}{2}\rho g U\left(\frac{b^2 - a^2}{c} + \frac{b'^2 - a'^2}{c'}\right). \tag{3.3}$$

Using (3.1)–(3.3) yields for the mean thrust $\overline{T}_b$ (positive in the direction of propagation, i.e. in the negative x_1 direction)

$$\begin{aligned}\overline{T}_b &= \overline{F}(x_1 = \infty) - \overline{F}(x_1 = -\infty) + U\overline{I} \\ &= \tfrac{1}{4}\rho g\left[\frac{2U + c}{c}(b^2 - a^2) + \frac{2U + c'}{c'}(b'^2 - a'^2)\right].\end{aligned} \tag{3.4}$$

On the other hand, the influx of energy at $x_1 = -\infty$ is found in Appendix B as

$$\overline{W}(x_1 = -\infty) = \tfrac{1}{4}\rho g(ca^2 + c'a'^2). \tag{3.5}$$

Similarly, one has for the outflux at $x_1 = \infty$

$$\overline{W}(x_1 = \infty) = \tfrac{1}{4}\rho g(cb^2 + c'b'^2). \tag{3.6}$$

The mean rate at which wave energy is being added is obtained from (B 14) as

$$U\overline{E} = \tfrac{1}{2}\rho g U(b^2 - a^2 + b'^2 - a'^2). \tag{3.7}$$

The mean rate of working by the thrust on the flow in the (x_1, z_1, t_1)-system is given by $-\overline{T}_b U$. Thus the power balance, using (3.4)–(3.7), may be written as

$$\begin{aligned}\overline{W}_b &= \overline{T}_b U + \overline{W}(x_1 = \infty) - \overline{W}(x_1 = -\infty) + U\overline{E} \\ &= \tfrac{1}{4}\rho g\left[\frac{(U + c)(2U + c)}{c}(b^2 - a^2) + \frac{(U + c')(2U + c')}{c'}(b'^2 - a'^2)\right].\end{aligned} \tag{3.8}$$

In the terminology of §2, we note that $\overline{T}_b$ and $\overline{W}_b$, as given by (3.4) and (3.8), are $O(\epsilon^2)$.

Inspection of the expressions (3.4) and (3.8) indicates that in two main cases the efficiency of propulsion is invariably of the same form as the efficiency obtained in §2 for slender bodies in waves. We distinguish two main cases:

(i) $$b'^2 - a'^2 = 0 \quad \text{with} \quad \eta = \frac{\overline{T}_b U}{\overline{W}_b} = \frac{U}{U + c} \quad (c > 0), \tag{3.9}$$

(ii) $$b^2 - a^2 = 0 \quad \text{with} \quad \eta = \frac{\overline{T}_b U}{\overline{W}_b} = \frac{U}{U + c'} \quad (c' < 0,\ 2U + c' \neq 0). \tag{3.10}$$

We shall now discuss some examples of the use of the expressions (3.4) and (3.8).

Propulsion by wavemaking

In case (i) one may assume that there is only one outgoing wave at $x_1 = \infty$ ($a = a' = b' = 0$ but $b \neq 0$) which is being generated by the body. From (3.4) and (3.8) we have

$$\left.\begin{aligned} \overline{T}_\mathrm{b} &= \tfrac{1}{4}\rho g \frac{2U+c}{c} b^2, \\ \overline{W}_\mathrm{b} &= \tfrac{1}{4}\rho g \frac{(U+c)(2U+c)}{c} b^2, \\ \eta &= \frac{\overline{T}_\mathrm{b} U}{\overline{W}_\mathrm{b}} = \frac{U}{U+c}. \end{aligned}\right\} \tag{3.11}$$

Thus (3.11) is valid for a body swimming in its own waves, and it is obvious that these results are also valid for bodies which are not completely submerged. The expressions (3.11) also apply to Longuet-Higgins' (1977) wavemaking boat with a Salter cam which oscillates at the stern and generates nearly two-dimensional surface waves. At a frequency of 3 Hz of the oscillator, his model was propelled at a speed $U = 0.12$ m/s. With $\omega_\mathrm{e}/2\pi = (U+c)/\lambda = 3$ Hz and $c^2 = g\lambda/2\pi$, we obtain $c = 0.625$ m/s and $\lambda = 0.25$ m. Using (3.11) yields an efficiency $\eta = 16\,\%$. We note that the mean thrust which balances the drag (frictional + wave resistance) is larger (38 % in the present example) with the result (3.11) than the thrust estimated from the outgoing wave upon neglecting the forward speed U.

On the other hand, one may assume, in case (ii), that there is only one incoming wave at $x_1 = \infty$ ($a = a' = b = 0$ but $b' \neq 0$) and that the body is moving steadily with this incoming wave, i.e. $U + c' = 0$. In this case one has

$$\left.\begin{aligned} \overline{T}_\mathrm{b} &= -\overline{D}_\mathrm{w} = -\tfrac{1}{4}\rho g b'^2, \\ \overline{W}_\mathrm{b} &= 0, \end{aligned}\right\} \tag{3.12}$$

which reproduces the classical result for the two-dimensional wave drag. Now combining the results (3.11) and (3.12) shows that the thrust generated by means of the outgoing waves precisely balances the 'steady wave drag' for

$$b^2 = \frac{cb'^2}{2U+c}, \tag{3.13}$$

while it is obvious that the net thrust by (3.11) is still being generated at an efficiency $\eta = U/(U+c)$. High efficiency is obtained with $c \ll U$. From the dispersion relation $2\pi c^2 = g\lambda$ it is clear that small values of c imply small values of λ. In view of the linearizations that were performed in the derivation of the expressions for $\overline{T}_\mathrm{b}$ and $\overline{W}_\mathrm{b}$ the steepness b/λ of the outgoing waves and thereby the admissible values of b and the corresponding level of the thrust which can be generated at high efficiency are restricted. It would be interesting to develop a nonlinear theory for these cases but this is beyond the scope of the present paper.

Critical frequency

In case (ii) with $b^2 - a^2 = 0$ and a following sea with $c' < 0$, we note that for $2U + c' = 0$ one has from (3.4) and (3.8)

$$\overline{T}_\mathrm{b} = 0, \quad \overline{W}_\mathrm{b} = 0 \quad \text{for } 2U + c' = 0. \tag{3.14}$$

This shows the relevance of the group velocity c'_g, which for deep-water waves is given by $c'_g = \frac{1}{2}c'$. When swimming at the group velocity of the following waves with $U = |c'_g|$, one has $2U + c' = 0$ and both $\overline{T}_b$ and $\overline{W}_b$ vanish. In this situation it turns out to be impossible to exchange energy and momentum $O(\epsilon^2)$ in the mean. We note that in the $O(\epsilon^3)$ and $O(\epsilon^4)$ slender-body results of §2 the group velocity of the waves did not come into the picture, since 'free-surface' effects were not included there.

The frequency of encounter ω_e when swimming at the group velocity is given by

$$\omega_e = k\,|U + c'| = kU, \tag{3.15}$$

which by virtue of the dispersion relation can also be expressed as

$$\omega_e = \frac{g}{4U}. \tag{3.16}$$

ω_e is known as a critical frequency in unsteady ship hydrodynamics. Not surprisingly, the frequency (3.16) turns out to be of critical importance in unsteady swimming problems. This can also be shown as follows:

With $U = 0$ and an incoming wave of amplitude a at $x_1 = -\infty$ but no incoming wave at $x_1 = \infty$ ($b' = 0$), the outgoing wave of amplitude a' at $x_1 = -\infty$ is the reflected wave and the outgoing wave of amplitude b at $x_1 = \infty$ is the transmitted wave. In this case one has $c' = -c$, and the power balance yields

$$\overline{W}_b = \tfrac{1}{4}\rho g(b^2 - a^2 + a'^2). \tag{3.17}$$

The force on the body in the direction of propagation of the incoming wave is

$$-\overline{T}_b = -\tfrac{1}{4}\rho g(b^2 - a^2 - a'^2). \tag{3.18}$$

In the case of power extraction one has $\overline{W}_b < 0$, and from (3.17) one has $b^2 < a^2 - a'^2$. Then with (3.18) one obtains $-\overline{T}_b > 0$, which means that with power extraction at $U = 0$ one always has a mean force on the power absorber in the direction of propagation of the incoming wave, as expected.

With $U \neq 0$ the frequency of encounter of the incident wave is

$$\omega_e = k\,|U + c|, \tag{3.19}$$

and the flow is periodic in a reference system moving steadily with the body. For the 'reflected' wave with wavenumber k' the dispersion relation now yields

$$\left(\frac{\omega_e}{k'} + U\right)^2 = \frac{g}{k'}, \tag{3.20}$$

which is a quadratic equation for k', of which the discriminant $g^2 - 4gU\omega_e$ vanishes for

$$\frac{\omega_e U}{g} = \frac{1}{4}, \tag{3.21}$$

which reproduces (3.16). This alternative derivation shows, however, that for

$$\frac{\omega_e U}{g} > \frac{1}{4} \tag{3.22}$$

there is no 'reflected' wave: one has $a' = 0$, unless of course there is a following sea from the start. It will be clear that when there is no incoming head sea ($a = 0$) the expression 'reflected', which might be appropriate whenever $\overline{W}_b < 0$ and $a \neq 0$, can be replaced by 'radiated'.

Forerunners can only exist when (3.22) is not satisfied. Inserting the numbers for

Longuet-Higgins's (1977) wavemaking boat in (3.16) and (3.22) indicates that the frequency of oscillation (3 Hz) was probably just below the critical value. Thus, in addition to some radiation to the sides, there may have been some forerunners. On the other hand, when (3.22) is satisfied, one always has perfect beaming for otherwise arbitrary oscillations of the body. All the energy and the momentum is being transmitted to the wake, and thrust is generated at an efficiency that is invariably given by $\eta = U/(U+c)$ for swimming oscillations at a frequency $\omega > g/4U$.

Weak perturbations of a head sea

In this example we suppose that an oncoming head sea of amplitude a at $x_1 = -\infty$ is being transmitted to a far-field wave behind the body of amplitude b with, from (B 16),

$$b^2 = a^2 + (\delta a)^2 + 2\delta a^2 \cos\psi. \tag{3.23}$$

For small values of $\delta \ll 1$ one thus obtains an interaction that is of relative order δ. It follows from the expressions (B 14) that there can be no such order-δ interaction with the oncoming wave ahead of the body for perturbations of relative order δ and negative phase velocity. This remains true even when the critical condition (3.22) is not satisfied. In this case one thus obtains, to leading order,

$$\overline{T}_{\mathrm{b}} = \tfrac{1}{2}\rho g \frac{2U+c}{c}\delta a^2 \cos\psi,$$

$$\overline{W}_{\mathrm{b}} = \tfrac{1}{2}\rho g \frac{(U+c)\,(2U+c)}{c}\delta a^2 \cos\psi, \tag{3.24}$$

with, invariably, $\eta = U/(U+c)$.

Owing to the presence of the oncoming wave, the leading-order terms in the momentum and energy flux perturbations are perfectly beamed. It is easily seen that similar conclusions apply in a following sea, except with $2U+c' = 0$.

We note that $\overline{T}_{\mathrm{b}}$ and $\overline{W}_{\mathrm{b}}$ in (3.24) are of interaction type, similar to the slender-body results of §2, in contrast with the other examples in §3.

The actuator-surface analogy

It is interesting to compare the efficiency of propulsion of an actuator body in waves,

$$\eta = \frac{U}{U+c} = \frac{1}{1+c/U}, \tag{3.25}$$

with the efficiency of an actuator surface with uniform loading in a uniform oncoming flow,

$$\eta = \frac{1}{1+a_1}, \tag{3.26}$$

where a_1 is the axial induction factor (see figure 3). It appears that c/U plays the role of the axial induction factor:

$$a_1 = \frac{c}{U}. \tag{3.27}$$

The analogy is not just a formal one. Upon replacing the vortex layer at the boundary of the wake far behind an actuator surface by a periodic row of discrete vortices, one obtains the same mean flow when the strength Γ of the vortices satisfies

$$\frac{\Gamma}{\lambda} = 2a_1 U, \tag{3.28}$$

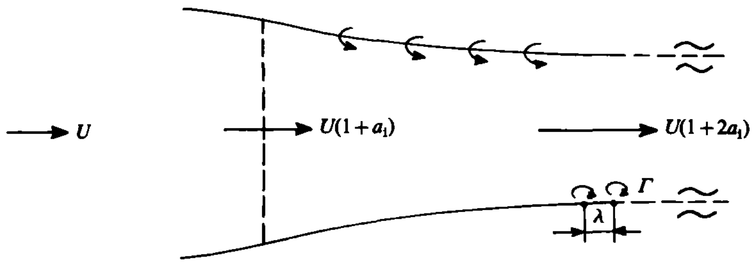

FIGURE 3. The vortex waves at the boundary of the wake far behind an actuator surface with constant loading are perfectly beamed with phase velocity $U+c = U(1+a_1)$.

where λ is the distance between two successive vortices. The rate of increase of kinetic energy in the flow due to discrete vortices of finite strength is infinite, but in taking the limit $\lambda \to 0$ one can require that Γ/λ remains constant and equal to $2a_1 U$. In the limit the rate of increase of kinetic energy (per unit volume) is finite. The vortices are transported at a velocity $a_1 U$ with respect to the water at rest outside the wake, and this velocity represents the phase velocity c of the travelling vortex wave

$$c = a_1 U, \tag{3.29}$$

which reproduces (3.27). Thus ideal propellers ($a_1 > 0$) and turbines ($a_1 < 0$) are found to behave like vortex wavemakers with perfect beaming.

The author is indebted to Professor J. A. Steketee for his continuous interest and support of the work described in this paper.

Appendix A. The mean thrust for flexible and deformable slender bodies in waves

The mean thrust may be expressed as

$$\overline{T} = \iint_S \overline{p\,\mathrm{d}y\,\mathrm{d}z} = \iint_S \overline{p\,\mathrm{d}Y\,\mathrm{d}Z} + \int_0^l \overline{L\frac{\partial h}{\partial X}}\,\mathrm{d}X. \tag{A 1}$$

The mean thrust has been discussed by Coene (1975) for flexible slender bodies with undeformable cross-sections, and the propulsion of the body was further evaluated on the assumption that the situation at the tail dominates the generation of thrust and the mean rate of working. At present we evaluate the contributions to (A 1) that do not depend on the situation at the tail and that are not associated with the shedding of vorticity. In the present context flexibility of the body refers to the motions $h(x,t)$ with respect to the stretched straight position. The deformability refers to the time dependence of the cross-sections, which includes the possibility of peristaltic deformations. The motions $h(x,t)$ as well as the deformations may be either passive or active.

The first term on the right-hand side of (A 1) can be evaluated by using Bernoulli's equation in the form (2.9). Neglecting free-surface effects, p_0 does not contribute to the mean thrust or drag. D'Alembert's paradox applies to bodies that deform periodically and do not shed vorticity. Since the kinetic energy in the flow varies periodically, the mean rate of working and the mean thrust or drag associated with p_0 vanish. The steady wave drag, however, also arising in the absence of an ambient wave, may be incorporated in the total drag which has to be balanced by the thrust. The drag or thrust associated with unsteady wavemaking is discussed in §3.

The part p_1 will contribute by virtue of the last term, involving the inclination of the lateral force only. From the first term of p_2, a contribution that is large $O(\epsilon^3)$ may arise:

$$\iint p_2 \,\mathrm{d}Y\,\mathrm{d}Z = -\rho\iint\left\{\left(\frac{\partial}{\partial T}+U\frac{\partial}{\partial X}\right)\Phi_w - w\frac{\partial\Phi_w}{\partial Z}+\frac{1}{2}\left(\frac{\partial\Phi_w}{\partial X}\right)^2+\frac{1}{2}\left(\frac{\partial\Phi_w}{\partial Z}\right)^2\right\}\mathrm{d}Y\,\mathrm{d}Z. \quad \text{(A 2)}$$

In calculating Φ_w in (A 2) one obtains for the value of Φ_w at $Z = 0$, in terms of magnitudes at the mean swimming depth, $z = 0$, i.e. $Z = -h(X, T)$,

$$\begin{aligned}\Phi_w(X,0,T) &= \Phi_w[X,-h(X,T),T]+h(X,T)\,\Phi_{wz}[X,-h(X,T),T]+\dots\\ &= \Phi_w^*+h\Phi_{wz}^*+\dots.\end{aligned} \quad \text{(A 3)}$$

Inserting (A 3) in (A 2) thus leads to contributions

$$\rho\int_0^l \overline{\frac{\partial S}{\partial X}\left(\frac{\partial}{\partial T}+U\frac{\partial}{\partial X}\right)\Phi_w^*}\,\mathrm{d}X+\rho\int_0^l\overline{\frac{\partial S}{\partial X}h\left(\frac{\partial}{\partial T}+U\frac{\partial}{\partial X}\right)\Phi_{wz}^*}\,\mathrm{d}X$$
$$-\tfrac{1}{2}\rho\int_0^l\overline{\frac{\partial S}{\partial X}(\Phi_{wX}^{*2}+\Phi_{wZ}^{*2})}\,\mathrm{d}X. \quad \text{(A 4)}$$

The first term in (A 4) is $O(\epsilon^3)$, and accounts for the case of steady wave riding with $U+c = 0$ and Φ_w^* independent of T. The first term also accounts for the effect of peristaltic deformations correlated with the oscillatory pressure due to the wave. The second term in (A 4), $O(\epsilon^4)$, due to the displacements h in the ambient wave, was erroneously omitted from Coene's (1975) paper. This was also pointed out by Coene (1977), who retained this term consistently for comparison with experiments. The third term in (A 4) vanishes since $\Phi_{wX}^{*2}+\Phi_{wZ}^{*2}$ is independent of X and T, while with the assumptions $S(0, T) = 0$ and $S(l, T) = 0$ the integral $\int_0^l (\partial S/\partial X)\,\mathrm{d}X$ vanishes for all T.

Rewriting (A 4) in the variables x, t and ϕ_w^* thus yields for the contribution of p_2 to the first term in (A 1)

$$\iint\overline{p_2\,\mathrm{d}Y\,\mathrm{d}Z} = \rho\int_0^l\overline{\frac{\partial S}{\partial x}\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\phi_w^*+h\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\phi_{wz}^*\right]}\,\mathrm{d}x, \quad \text{(A 5)}$$

which includes the $O(\epsilon^3)$ and $O(\epsilon^4)$ terms. Equation (A 5) generalizes and supersedes (A 2) in the Appendix of Coene (1975). It is clear from (A 5) that the $O(\epsilon^3)$ term vanishes in the mean for undeformable sections and oscillatory ϕ_w^*. It should be noted that (A 5) is not the complete contribution due to p_2, since the second term in (A 1) contains another $O(\epsilon^4)$ contribution due to p_2 by virtue of the inclination of the lateral force, which will be accounted for below.

The contribution of p_3 to the first term on the right-hand side of (A 1) is

$$\iint_S\overline{p_3\,\mathrm{d}Y\,\mathrm{d}Z} = \rho\iint_S\overline{\left\{w^*\frac{\partial\Psi}{\partial Z}-\frac{1}{2}\left(\frac{\partial\Psi}{\partial Y}\right)^2-\frac{1}{2}\left(\frac{\partial\Psi}{\partial Z}\right)^2\right\}}\,\mathrm{d}Y\,\mathrm{d}Z. \quad \text{(A 6)}$$

The integrand in (A 6), for certain values of X and T, is formally equivalent to the pressure distribution due to the motion of a cylinder with velocity w^* in the Z-direction through water at rest. By virtue of (2.5) one may rewrite (A 6), to leading order, as

$$\rho\iint_S\overline{w^{*2}\left\{\frac{\partial\phi}{\partial Z}-\frac{1}{2}\left(\frac{\partial\phi}{\partial Y}\right)^2-\frac{1}{2}\left(\frac{\partial\phi}{\partial Z}\right)^2\right\}}\,\mathrm{d}Y\,\mathrm{d}Z. \quad \text{(A 7)}$$

Equation (A 7) can be evaluated (to $O(\epsilon^4)$) upon introduction of the system (X_1, Y_1, Z_1, T_1), with

$$X_1 = X-UT, \quad Y_1 = Y, \quad Z_1 = Z, \quad T_1 = T. \quad \text{(A 8)}$$

The kinetic energy of the water per unit length in the X_1 direction is $\frac{1}{2}\rho A w^{*2}$. A time δT_1 later the kinetic energy has changed by an amount $\frac{1}{2}\rho[\partial(Aw^{*2})/\partial T_1]\,\delta T_1$, which is equal to the amount of work done by the body against the pressure forces in the water:

$$\tfrac{1}{2}\rho\frac{\partial}{\partial T_1}(Aw^{*2})\,\delta T_1 = \rho w^{*}\frac{\partial}{\partial T_1}(Aw^{*})\,\delta T_1 - \tfrac{1}{2}\rho w^{*2}\frac{\partial A}{\partial T_1}\,\delta T_1. \tag{A 9}$$

The first term on the right-hand side of (A 9) is the amount of work done in the Z_1 direction. The second term represents the amount of work done by the thrust in the X_1 direction and the work done by the peristaltic deformations. With $\partial/\partial T_1 = \partial/\partial T + U\partial/\partial X$ one thus obtains for the thrust, from p_3, per unit length,

$$\frac{\partial T_{p3}}{\partial X} = \tfrac{1}{2}\rho w^{*2}\frac{\partial A}{\partial X}. \tag{A 10}$$

Adding the second term in (A 1), using (2.13), in the variables x, t and ϕ^*_w,

$$\int_0^l \overline{L\frac{\partial h}{\partial x}} = -\rho\int_0^l \overline{\frac{\partial h}{\partial x}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)(w^*A)}\,\mathrm{d}x + \rho\int_0^l \overline{S\frac{\partial h}{\partial x}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\phi^*_{wz}}\,\mathrm{d}x, \tag{A 11}$$

the total mean thrust can be expressed as $\{(\text{A } 5)+(\text{A } 10)+(\text{A } 11)\}$:

$$\begin{aligned}\overline{T} = \rho\int_0^l &\overline{\frac{\partial S}{\partial x}\left\{\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\phi^*_w + h\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\phi^*_{wz}\right\}}\,\mathrm{d}x \\ &+\tfrac{1}{2}\rho\int_0^l \overline{w^{*2}\frac{\partial A}{\partial x}}\,\mathrm{d}x - \rho\int_0^l \overline{\frac{\partial h}{\partial x}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)(w^*A)}\,\mathrm{d}x \\ &+\rho\int_0^l \overline{S\frac{\partial h}{\partial x}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\phi^*_{wz}}\,\mathrm{d}x + O(\epsilon^5),\end{aligned} \tag{A 12}$$

which generalizes and supersedes (A 6) and (3.10) of Coene (1975). Upon manipulation of (A 12) with the assumptions $S(0) = 0$, $S(l) = 0$, $A(0) = 0$, $A(l) \neq 0$, the mean thrust may be rewritten as

$$\begin{aligned}\overline{T} = \rho\int_0^l &\overline{(\phi^*_{wx}+h\phi^*_{wzx})\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)S}\,\mathrm{d}x + \rho\int_0^l \overline{(A+S)\,w^*\phi^*_{wzx}}\,\mathrm{d}x \\ &+\tfrac{1}{2}\rho\left[\overline{A\left\{\left(\frac{\partial h}{\partial t}-\phi^*_{wz}\right)^2 - U^2\left(\frac{\partial h}{\partial x}\right)^2\right\}}\right]_{x=l},\end{aligned} \tag{A 13}$$

which generalizes and supersedes (3.11) of Coene (1975).

Appendix B. Some properties of superimposed deep-water waves

A two-dimensional sinusoidal wave on deep water of amplitude a can be expressed as

$$\phi = ac\,\mathrm{e}^{kz}\cos\{k(x-ct)\}. \tag{B 1}$$

In (B 1) c is the phase velocity, and the wavenumber k is related to the wavelength λ by $k = 2\pi\lambda^{-1}$. The water surface is given by

$$\zeta = -a\sin\{k(x-ct)\}. \tag{B 2}$$

For deep-water waves the dispersion relation can be expressed as

$$c^2k = g. \tag{B 3}$$

We now evaluate some properties that are relevant to propulsion problems.

The time-averaged horizontal component of momentum per unit surface is given by

$$\bar{I} = \frac{kc}{2\pi}\int_0^{2\pi/kc}\int_{-\infty}^{\zeta}\rho\phi_x\,\mathrm{d}z\,\mathrm{d}t = -\rho g^{-1}\overline{(\phi_x\phi_t)}_{z=0} = \tfrac{1}{2}\rho g c^{-1}a^2. \tag{B 4}$$

The excess potential energy due to the wave per unit horizontal area is given by

$$\bar{V} = \rho g\overline{\int_0^{\zeta} z\,\mathrm{d}z} = \tfrac{1}{4}\rho g a^2. \tag{B 5}$$

The kinetic energy in the waves per unit horizontal area has the same magnitude:

$$\bar{K} = \tfrac{1}{2}\rho\overline{(\phi\phi_z)}_{z=0} = \tfrac{1}{4}\rho g a^2. \tag{B 6}$$

Thus for the total energy per unit surface one has the well-known result

$$\bar{E} = \bar{V}+\bar{K} = \tfrac{1}{2}\rho g a^2. \tag{B 7}$$

The mean flux of horizontal momentum across a vertical plane x = const per unit length parallel to the crests of the waves may be expressed as

$$\bar{F}_0+\bar{F} = \overline{\int_{-\infty}^{\zeta}(p+\rho\phi_x^2)\,\mathrm{d}z}, \tag{B 8}$$

where $\bar{F}_0$ in the flux in the absence of waves. Subtracting $\bar{F}_0$, we obtain the mean excess flux due to the waves (with $p = p_0+p_\mathrm{e} = \rho g z+p_\mathrm{e}$):

$$\bar{F} = \overline{\int_{-\infty}^{\zeta}(p_\mathrm{e}+\rho\phi_x^2)\,\mathrm{d}z}-\overline{\int_0^{\zeta}\rho g z\,\mathrm{d}z}. \tag{B 9}$$

From Bernoulli's equation we have

$$p_\mathrm{e} = -\tfrac{1}{2}\rho(\phi_x^2+\phi_z^2)-\rho\phi_t. \tag{B 10}$$

With $-g\zeta = (\phi_t)_{z=0}$, one obtains to second order:

$$\bar{F} = \tfrac{1}{2}\rho g^{-1}\overline{(\phi_t^2)}_{z=0} = \tfrac{1}{4}\rho g a^2. \tag{B 11}$$

The mean value of the energy flux across a vertical plane x = const. per unit length in the direction of the crests of the waves is given, to second order, by

$$\bar{W} = \int_{-\infty}^{0}\overline{p_\mathrm{e}\phi_x}\,\mathrm{d}z = -\rho(2k)^{-1}\overline{(\phi_t\phi_x)}_{z=0} = \tfrac{1}{4}\rho g a^2 c. \tag{B 12}$$

We now consider a superposition of two waves $\phi_{1+2} = \phi_1+\phi_2$, and we distinguish two cases, (i) and (ii).

(i) $c_1 \neq c_2$ with, at $z = 0$,

$$\left.\begin{aligned}\phi_1 &= a_1c_1\cos\{k_1(x-c_1t)\},\\ \phi_2 &= a_2c_2\cos\{k_2(x-c_2t)\}.\end{aligned}\right\} \tag{B 13}$$

Upon replacing the time average per period by the long-time average, the bilinear quantities evaluated above for a single wave are readily found for the combined flow:

$$\left.\begin{aligned}\bar{I}_{1+2} &= \tfrac{1}{2}\rho g(c_1^{-1}a_1^2+c_2^{-1}a_2^2),\\ \bar{E}_{1+2} &= \tfrac{1}{2}\rho g(a_1^2+a_2^2),\\ \bar{F}_{1+2} &= \tfrac{1}{4}\rho g(a_1^2+a_2^2),\\ \bar{W}_{1+2} &= \tfrac{1}{4}\rho g(c_1a_1^2+c_2a_2^2).\end{aligned}\right\} \tag{B 14}$$

(ii) $c_1 = c_2 = c$, with, at $z = 0$,

$$\left.\begin{aligned}\phi_1 &= a_1 c \cos\{k(x-ct)\},\\ \phi_2 &= a_2 c \cos\{k(x-ct)+\psi\}.\end{aligned}\right\} \tag{B 15}$$

It is easily verified that (B 15) combine to give a simple wave with the same phase velocity c. The amplitude a is given by

$$a^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos\psi, \tag{B 16}$$

and the expressions obtained for single waves remain valid. It follows that only in case (ii) with $c_1 = c_2 = c$ does one have interaction whenever $\cos\psi \neq 0$.

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